Quasi-Explicit Calibration of Gatheral's SVI model

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Abstract

We present a procedure - based on dimension reduction in parameters space - providing a quasi-explicit calibration of J. Gatheral's SVI model for implied variance. The resulting parameter identification is reliable and stable.

1 A simple model and a delicate calibration

Jim Gatheral's SVI model [1] describes implied variance with the following parametric form:

$$v(x) = \sigma_{BS}^2(x) = a + b\Big(\rho(x-m) + \sqrt{(x-m)^2+\sigma^2}\Big), \tag{1}$$

where v is the implied variance of market prices of Vanillas at fixed time-to-maturity T, x the log-forward moneyness $x = log(K/F_T)$ and a, b, ρ, m, σ are the model parameters. Parametric models (e.g. SVI or the functional forms obtained by Taylor's approximation in CEV or SABR models) are of common use in the treatment of the volatility surface. Apart from the extrapolation of smile points, they provide a smooth representation of the market smile and consequently facilitate the calibration of stochastic models for the underlying (including the reconstruction of a local volatility surface via Dupire's formula, for which interpolation in time must also be taken into account). It is well known that the SVI parametric form (1) proves to have outstanding calibration performances to single-maturity slices of the implied smile on many Equity indexes. Nevertheless, it is also common knowledge that the least-square calibration of (1) is typically affected by the presence of several local minima. To our experience, even when SVI parameterization is calibrated to simulated data, i.e. a smile produced by SVI itself, local minima that are difficult to sort out (least square objective $\approx 10^{-8}$ for reasonable volatility values, $\sqrt{v} \approx 20\% - 40\%$) are found far away from the global one (objective $= 0$).

This unpleasant feature tends to bring some difficulties if one wants to design a parameter identification strategy for SVI model which is sufficiently robust and stable. The solution yield by a least square optimizer usually has a strong dependence on the input starting point. Then, smart initial guesses of parameter sets can be made by looking to the 'geometry' of the observed smile (asymptotic slopes, minimum value), and on the other hand the calibration can be restarted from several different initial guesses and/or using more than one non-linear optimizer. Nevertheless, usual strategies to find the initial guess are not defect-free and require attention, since the desired smile features are not available in all the cases and often not for all maturities (e.g. the wings are not both observed or the smile has no clearly visible minimum). The calibration reset, though useful, still does not guarantee that one manages to overcome all the local minima and may anyhow lead to ambiguous choices of optimal parameters, since the same smile can be - remarkably well - calibrated with sets of parameters that are totally different one from the other. The big issue, then, is the stability of calibrated parameters with respect to time-to-maturity. This is a feature which comes into play in a significant way when trying to parameterize the whole volatility surface.

This document presents a procedure providing a trustworthy and stable calibration of SVI parametric form (1), which has the pleasant feature of not being strongly sensitive to initial parameter guess. We rely on some simple observations on the symmetries of the functional form (1) to downsize the minimization problem from dimension 5 (the number of parameters in (1)) to dimension 2 (namely, m and σ), while the optimization over the remaining 3 is performed explicitly. Last but not least, the method yields an optimal parameter set which is automatically consistent with the arbitrage constraint on the slopes of implied variance. The procedure for the Quasi-Explicit calibration is presented in section 3, while section 2 discusses the constraints that are introduced in the parameters space.

2 Parameter constraints and limiting cases

The parameters a, b, ρ, m, σ in general depend on time-to-maturity T. We assume that b, σ, ρ satisfy

$$b > 0, \;\; \sigma \geq 0, \;\; \rho \in [-1, 1].$$

Further conditions on b and ρ follow from well known arbitrage conditions (cf section 2.2). In addition, we will discuss some constraints on parameters σ and a which are related to the well-posedness of the calibration problem (section 2.3). We recall once again that in the current document we just look to the parameterization of time-slices of implied variance.

2.1 Slopes and minimum

We review the main interesting properties of the parametric form (1). The left and right asymptotes are respectively (cf.[1])

$$\begin{aligned} v_L(x) &= a - b(1-\rho)(x-m), \\ v_R(x) &= a + b(1+\rho)(x-m). \end{aligned}$$

The term adding to a in (1) is always positive and convex w.r.t.x. v has a unique minimum point if $\rho^2 \neq 1$, in particular:

- if $\rho^2 \neq 1$, the minimum is $a + b\sigma\sqrt{1-\rho^2}$ attained at $x^* = m - \frac{\rho\sigma}{\sqrt{1-\rho^2}}$;
- if $\rho^2 = 1$, v is non-increasing for $\rho = -1$ and non-decreasing for $\rho = 1$ and
  - if $\sigma \neq 0$, v is strictly monotone and the minimum is never attained (nevertheless, $v \to a$ for very positive or very negative x);
  - if $\sigma = 0$, v has the shape of a Put or Call payoff of strike m (v is worth a for $x \geq m$ if $\rho = -1$ and for $x \leq m$ if $\rho = 1$).

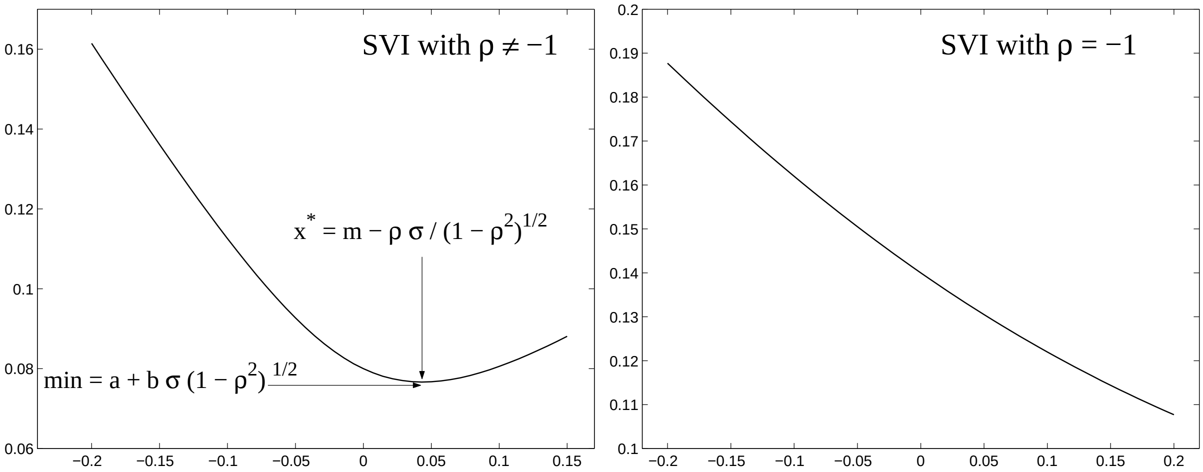

Figure 1: Examples of SVI smile shapes. SVI parameters are $a = 0.04, b = 0.4, \rho = -0.4, m = 0.05, \sigma = 0.1$ (left) and $a = 0.04, b = 0.2, \rho = -1, m = 0.1, \sigma = 0.5$ (right).

2.2 Arbitrage constraints (b and ρ)

A necessary condition for the absence of arbitrage is a constraint on the maximal slopes of total implied variance $Tv(x)$. As found in [2], this condition reads

$$\forall\, x, \forall\, T, \quad |T\partial_x v(x)| \leq 4. \tag{2}$$

As stated in [1], this translates into the following equivalent condition on b and ρ :

$$b \leq \frac{4}{(1+|\rho|)T}. \tag{3}$$

2.3 Limiting cases $\sigma \to 0$ and $\sigma \to \infty$ (almost-affine smiles)

As observed in section 2.1 in the case $\rho^2 = 1$, letting $\sigma \to 0$ gives a piecewise affine parameterisation of variance. In the two regions $x < m$ and $x > m$, variance reads respectively

$$v(x) = a + b(\rho \mp 1)(x - m). \tag{4}$$

Smiles which can be excellently fitted with an affine (monotone) parameterisation $v(x) = px + q$ (we will refer to these as to "almost-affine" smiles) are not uncommon on Equity indexes, in particular for large maturities. Clearly, the calibration of SVI model to an almost-affine smile is an ill-posed problem, in the sense that there exists infinitely many solutions to the minimization problem. Indeed, if we think of a downward smile to fix ideas, it is sufficient to let $\sigma \to 0$ and take m to be greater than the largest observed log-moneyness (to pick the minus sign in (4)) and the matching of the two relevant quantities, i.e. smile slope p and intercept q, yields the two equations

$$\begin{aligned} b(\rho-1) &= p \\ a - bm(\rho-1) &= q, \end{aligned}$$

corresponding to infinitely many choices of the parameters a, b, ρ.

The same kind of limiting behaviour is attained in the limit $\sigma \to \infty$ and $a \to -\infty$, in the precise way specified as follows. *A priori*, indeed, negative values of a could be allowed, since the positivity of parameterisation (1) is simply achieved by asking that the minimum of v (when attained) be non-negative, i.e.

$$a \geq -b\sigma\sqrt{1-\rho^2}$$

(if the minimum is not attained, then $\rho^2 = 1$ and the condition becomes $a \geq 0$).
Assume then $a < 0$ and $\sigma >> 1$, so that

$$\begin{aligned} v(x) &= a + b\Big(\rho(x-m) + \sqrt{\sigma^2 + (x-m)^2}\Big) \\ &= -|a| + b\rho(x-m) + b\sigma\sqrt{1 + \frac{(x-m)^2}{\sigma^2}} \\ &\sim -|a| + b\rho(x-m) + b\sigma\Big(1 + \frac{(x-m)^2}{2\sigma^2}\Big) \\ &\sim_{|a|=b\sigma}\ b\rho(x-m) + b\frac{(x-m)^2}{2\sigma}. \end{aligned}$$

Hence

$$\lim_{\sigma\to\infty, a\to-\infty, |a|=b\sigma} v(x) = b\rho(x-m)$$

for any value of x, and this correspond again to an affine smile whose slope and intercept identify the product $b\rho$ and the parameter m, but not b, ρ and m separately.

Smiles tend to flatten with increasing time to maturity, and curved smiles can continuously deform into almost-affine ones. Since the stability of the calibration is the features we have in mind, we would like the calibration strategy to avoid falling into the instable behaviour caused by limiting SVI cases. Hence, we restrict ourselves to the situation where:

$$\begin{aligned} &\sigma \geq \sigma_{\min} > 0, \\ &a \geq 0. \end{aligned} \tag{5}$$

We set the positive lower bound $\sigma_{\min}$ for σ ($\sigma_{\min} = 0.005$ in our numerical examples) in the sense that we state that if this threshold is reached, then an unambiguous calibration of SVI is not doable, i.e. any precise choice of model parameters is arbitrary (of course one can decide, for example, to inherit one of the SVI parameters from the ones calibrated to the previous time-slice - if any - but this goes back to user choices). In the

same way, we constrain a to positive values to avoid the $\sigma \to \infty$, $a \to -\infty$ limiting behaviour. Setting an upper bound for σ would prevent it from assuming too high values, but this would not avoid the phenomena of "coupling" of low values of a and high values of σ, where σ sticks to $\sigma_{\max}$ and a becomes very negative.
We just add here that an obvious upper bound on a is:

$$a \leq \max_i \{v_i\}, \tag{6}$$

where the v_i's are the observed variances at the given maturity. Condition (6) simply follows from a consistent vertical location of the graph of (1): clearly the curve v giving the optimal fit cannot be systematically greater than the largest observed variance.

3 Indeed, not more than a linear problem

As it stands, the calibration of SVI parametric form cast as a least square problem yields an optimization problem in dimension 5. We show that, relying on some simple observations on the properties of the functional form (1), one can reformulate the problem reducing the main dimension from 5 to 2.

3.1 Dimension reduction: drawing out the linear objective

We focus hereafter on the total variance $\tilde{v} = Tv$ rather than on variance. The main ingredient of the method is the fact that, by means of the change of variables

$$y = \frac{x - m}{\sigma}$$

the SVI parametrization transforms into

$$\tilde{v}(y) = aT + b\sigma T(\rho y + \sqrt{y^2 + 1}).$$

This expression nicely shows how, for fixed values of m and σ, the support of the curve Tv is fully determined by a, ρ and the product $b\sigma$. Thus, most important, if we redefine the parameters as

$$\begin{aligned} c &= b\sigma T \\ d &= \rho b\sigma T \\ \tilde{a} &= aT, \end{aligned}$$

then $\tilde{v}(y)$ turns out to depend linearly on c, d, a:

$$\tilde{v}(y) = \tilde{a} + dy + c\sqrt{y^2 + 1}.$$

Therefore, for fixed m and σ, we look for the solution of the problem:

$$(P_{m,\sigma}) \qquad \min_{(c,d,\tilde{a})\in D} f_{\{y_i,v_i\}}(c,d,\tilde{a})$$

where $f_{\{y_i,v_i\}}$ is the cost function

$$f_{\{y_i,v_i\}}(c,d,\tilde{a}) = f(c,d,\tilde{a}) = \sum_{i=1}^{n}\left(\tilde{a} + dy_i + c\sqrt{y_i^2+1} - \tilde{v}_i\right)^2,$$

with $\tilde{v}_i = Tv_i$, and D is the compact and convex domain (a parallelepiped)

$$D = \begin{cases} 0 \le c \le 4\sigma \\ |d| \le c \text{ and } |d| \le 4\sigma - c \\ 0 \le \tilde{a} \le \max_i\{\tilde{v}_i\} \end{cases}$$

which is obtained from bounds (3) and (5)-(6) on parameters b, ρ and a.
Letting $(c^*, d^*, \tilde{a}^*)$ denote the solution of $P_{m,\sigma}$ and (a^*, b^*, ρ^*) the corresponding triplet for a, b, ρ, then the complete calibration problem is restored as

$$(P) \quad \min_{m,\sigma} \sum_{i=1}^{n}(v_{m,\sigma,a^*,b^*,\rho^*}(x_i) - v_i)^2.$$

Our goal is therefore to solve $P_{m,\sigma}$ in the fastest and most accurate way: once this is done, the only task left is to look for the solution of the 2-dim problem P.

3.2 Explicit solution of the reduced problem

$P_{m,\sigma}$ (the reduced problem) is a convex optimization problem with linear program, and *all* the constraints defining the admissible domain D are linear. It is clearly seen, then, that this problem admits an explicit solution, and becomes extremely easy to deal with.

Since the cost function f is convex, differentiable and its gradient is zero at just one point (if the target smile contains at least three different points!), only two scenarios are possible:

- the minimum of f over D is attained at the interior of D, and this is the global minimum of f;
- the minimum of f over D is attained on the boundary ∂D.

Then, this yields the simple recipe:

Step 1. find the global minimizer of f, solving the *linear* system $\nabla f = 0$. If the output belongs to D, then stop;

Step 2. if *Step 1* yields a global minimum outside D, then look for $\min_{\partial D} f$.

The constrained optimization problem addressed in *Step 2* can be solved applying a Lagrange multipliers method for each one of the sides of the domain D. Then, *Step 2* involves only the solution of 3×3 linear systems (plus a few explicit one-dimensional minimizations, along the perimeter of the sides).
Once the two steps are accomplished, the solution of the reduced problem $(P_{m,\sigma})$ is achieved in explicit form. The calibration of (1) is then carried out solving the 2-dimensional problem (P) with some iterative optimizer, whose performances will be extremely enhanced with respect to the original problem in full dimension.

4 Numerical Results

We display some numerical tests and calibration results showing the performances of the Quasi-Explicit method.
Table 1 compares the "direct" procedure, i.e. standard least square calibration in dimension 5, and the Quasi-Explicit method for a SVI model calibrated to simulated data, i.e. a smile generated by SVI itself, for fixed $T = 1$. RMSE is $\sqrt{\sum_i (v(x_i) - v_i)^2}$, hence when looking to RMSE values one must take into account that the natural scale is the one of a variance. For the Standard Least Square calibration, the input value for a is inferred from the minimum observed variance, and the calibration is restarted 10 times from 10 randomly chosen points ($b \in [0, 0.5], \rho \in [-1, 1], m \in [2\min(x_i) < 0, 2\max(x_i) > 0], \sigma \in [0, 1]$). We do not go to great effort here in identifying a smart initial guess for all the parameters since the randomized procedure works quite well anyway, and our intention is rather to display the performances of the Quasi-Explicit calibration. We recall that, for the latter, no inputs for a, b and σ are needed; moreover, we take the initial guesses for m and σ as simple as it might be, i.e. a randomly chosen point. Standard Least Square optimization is performed with truncated-Newton algorithm, while the optimization over m and σ for the Quasi-Explicit method employs Nelder-Mead simplex algorithm. As it is seen from Table 1, even if a classical calibration can work properly, the Quasi-Explicit technique brings the objective to extremely small values. Moreover, the calibration we have obtained in the case $\rho = -0.9$ finely shows how a downward SVI smile can be more than reasonably calibrated with a SVI smile reaching its minimum (and then pointing upwards) for large values of the log-moneyness, hence with a set of parameters which is far away from the true one.
Table 2 and Figure 2 display the result of the Quasi-Explicit calibration of SVI model to the market-implied smile on DAX and EuroStoxx 50 indexes, for two different dates (20 August and 22 September 2008, respectively). Concerning Table 2, the quality of the fit is excellent through all maturities, somehow worse just for the very shortest

Method	Parameters	a	b	ρ	m	σ	RMSE
	True value	0.04	0.1	-0.5	0.0	0.1	
Standard LS	start. pt	0.048	10 r.p.	10 r.p.	10 r.p.	10 r.p.	
	calibrated	0.041	0.098	-0.51	$-4e^{-4}$	0.095	$7.9e^{-8}$
Quasi-Expl.	start. pt	-	-	-	1 r.p.	1 r.p.	
	calibrated	0.040	0.10	-0.50	$-8e^{-7}$	0.100	$5.0e^{-14}$
	True value	0.1	0.06	-0.9	0.24	0.06	
Standard LS	start. pt	0.11	10 r.p.	10 r.p.	10 r.p.	10 r.p.	
	calibrated	0.004	0.11	-0.19	0.73	0.58	$8.3e^{-8}$
Quasi-Expl.	start. pt	-	-	-	1 r.p.	1 r.p.	
	calibrated	0.10	0.060	-0.90	0.240	0.060	$3.4e^{-17}$

Table 1: Calibration of SVI model to simulated data (r.p. = random points). The two calibration strategies, Standard Least Square ($dim = 5$) and the Quasi-Explicit method are compared, for $T = 1$. For Standard Least Square, the starting value for a is inferred from minimum variance.

one, $T = 1$ month. Calibrated parameters show a good stability: obviously, a dependence w.r.t. time to maturity is expected - time-dependence is not taken into account by SVI model - but the important fact here is that the parameters do not show a noisy behaviour (a smooth time-dependence is particularly seen for a and b). Concerning the behaviour of ρ: ρ is different than -1 just for the first time-slice, which is the only non downward-pointing smile (cf. Figure 2, the smile on DAX on 20 August shows the same feature), and then it sticks to -1, because for all other maturities the smile is purely decreasing. The Quasi-Explicit method has indeed the tendency to fit decreasing smiles with anti-correlated SVI parameterizations.

T (Yrs)	a	b	ρ	m	σ	RMSE
0.082	0.027	0.234	0.068	0.100	0.028	$1.6e^{-6}$
0.16	0.030	0.125	-1.0	0.074	0.050	$2.8e^{-7}$
0.26	0.032	0.094	-1.0	0.093	0.041	$2.1e^{-7}$
0.33	0.028	0.105	-1.0	0.096	0.072	$1.3e^{-7}$
0.58	0.026	0.080	-1.0	0.127	0.098	$7.1e^{-8}$
0.83	0.026	0.066	-1.0	0.153	0.113	$1.8e^{-8}$
1.33	0.031	0.047	-1.0	0.171	0.065	$5.2e^{-8}$
1.83	0.037	0.039	-1.0	0.152	0.030	$9.1e^{-10}$
2.33	0.036	0.036	-1.0	0.200	0.083	$1.3e^{-9}$
2.82	0.038	0.036	-1.0	0.170	0.139	$2.4e^{-9}$
3.32	0.034	0.032	-1.0	0.246	0.199	$7.2e^{-10}$
4.34	0.044	0.028	-1.0	0.188	0.069	$2.6e^{-7}$

Table 2: Calibration of SVI model to the implied smile on the DAX Index on 20 August 2008. Each maturity is separately calibrated.

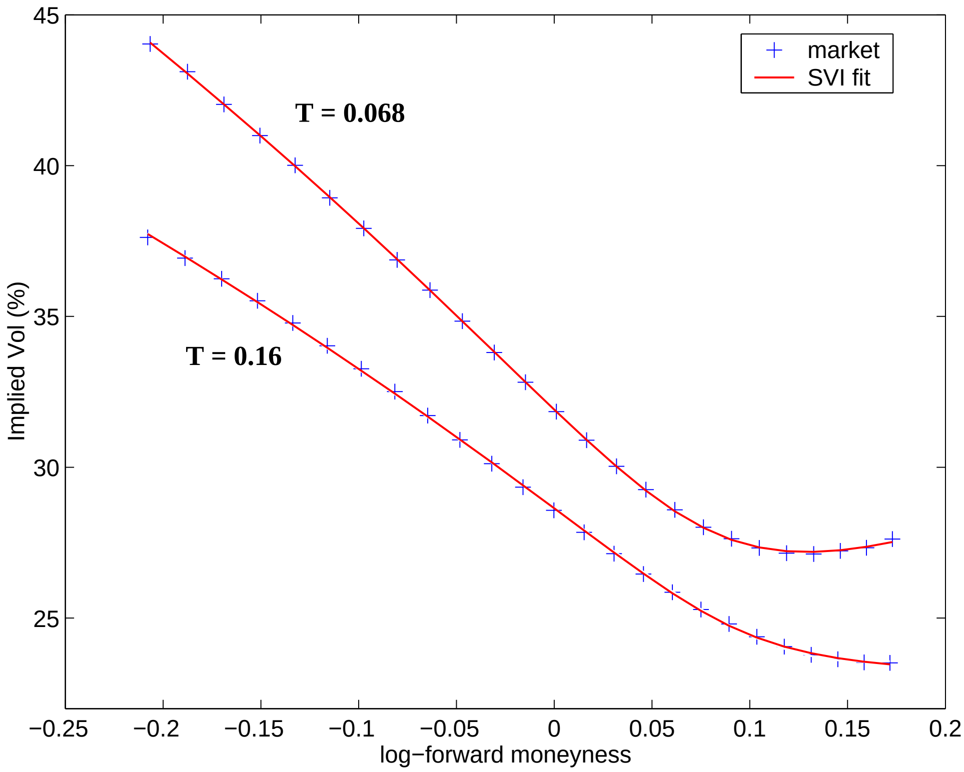

Figure 2: Calibration of SVI model to the implied smile on the EuroStoxx 50 Index on 22 September 2008, for the two shortest maturities.

5 Conclusions

Given the excellent performances of the Quasi-Explicit method - at least for what concerns calibration on Equity indexes - we claim that this methodology responds properly to the question of how to obtain an unambiguous identification of a time-slice of implied variance in terms of a set of SVI parameters. Once this high-quality fit is achieved, the SVI functional form can serve in many ways. Besides smile point extrapolation, one can recast the calibration of any stochastic model for the underlying as a calibration to the smooth objective (1). The matching of the geometry (levels, slopes and curvature) of the two model smiles can lead to explicit mappings of SVI parameters onto the ones of the chosen model, in the spirit (in another context) of the calibration methodology of [3]. Of course, this subject includes the issue of extracting a local volatility surface with Dupire's formula: since this operation needs interpolation in time, at this level the time-interpolation mechanism becomes as well a crucial point.

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