Model Validation: theory, practice and perspectives

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Contents

In July of 2009, the Basel Committee on Banking Supervision issued a directive [21] requiring that financial institutions quantify model risk. The Committee further stated that two types of risks should be taken into account: "The model risk associated with using a possibly incorrect valuation, and the risk associated with using unobservable calibration parameters". The resulting adjustments must impact Tier I regulatory capital, and the directive must be implemented by the end of 2010.

On the surface, this seems to be a simple adjustment to the market risk framework, adding model risk to other sources of risk that have already been identified within Basel II. In fact, quantifying model risk is much more complex because the source of risk (using an inadequate model) is much harder to characterize. Twelve months away from the deadline, there is no consensus on this topic. There is fortunately a growing body of literature, both from the academia and the industry, and the purpose of this paper is to summarize the development of the notion of "model risk" and present the current state of the art, before outlining open issues that must be resolved in order to define a consistent framework for measuring model risk.

The Nature of Model Risk

Financial assets can be divided into two categories. In the first category, we find the assets for which a price can be directly observed in the financial market place. These are the liquid assets for which there are either organized markets (e.g. Futures exchanges) or a liquid OTC market (e.g. interest rate swaps). For the vast majority of assets, however, price cannot be directly observed, but needs to be inferred from observable prices of related instruments. This is typically the case for financial derivatives whose price is related to various features of the *primary* assets, depending on a model. This process is known as "marking-to-model", and involves both a mathematical algorithm and subjective components, thus exposing the process to estimation error. Beyond this generic observation, the notion of Model Risk has been interpreted in at least three manners.

- A first interpretation of Model Risk stems from the observation that various models can be calibrated to perfectly price a set of liquid instruments, but produce inconsistent estimates for an exotic product. If one accepts that there is one "true" model, then model risk refers to the risk of mis-specification.
- A second interpretation focuses on the operational use of a model, that is used not only to compute a price, but equally importantly to compute risk indicators for dynamic hedging. In a perfect world where the true process for each risk factor is known, and where hedge instruments are readily available, we know that to

each derivative corresponds a dynamic replicating portfolio. Thus “model-risk” can be assessed by observing the hedging error, i.e. the discrepancy between the payoff of a derivative and the value of the replicating portfolio.

- Well publicized events in the banking industry have highlighted a third manifestation of Model Risk. When liquidity is low, how should a product be “marked to model” so as to minimize the risk of discrepancy when a transaction can finally be observed? To quote Rebonato [23]:

  > Model risk is the risk of occurrence of a significant difference between the mark-to-model value of a complex and/or illiquid instrument, and the price at which the same instrument is revealed to have traded in the market.

In summary, Model Risk can lead to both mis-pricing and mis-management of the hedging strategy. Mis-pricing clearly will have the most spectacular consequences, but mis-hedging is an equally serious issue.

The importance of Model Risk has gradually emerged in the industry: in first-tier banks, *Model Validation* teams have been created as an independent power besides front-office trading/quant teams. These organizations are now briefly reviewed.

Model Validation teams

These teams are usually organized by asset class (equity, commodities, FX, fixed income in a broad sense, credit, cross-currency). The day-to-day duty of all these model validation teams is to qualitatively and quantitatively validate the models used by the front-office. A closer look, however, reveals that their roles vary widely across institutions:

- In its narrowest interpretation, model validation is an almost pure theoretical and qualitative matter, which requires a solid mathematical finance background and a good sense for the business domain. It focuses on whether a model or methodology is adequate for the purpose it is used for by the front-office:
  - do not use local volatility models for pricing Cliquet options since they will be sensitive to the forward volatility which the local volatility models do not encompass,
  - do not use the SABR time-slice approximation for small strikes because it will yield artificial arbitrage opportunities.

- In a broader definition, the role involves the validation not only of the models or methodologies at hand, but also of their *implementations*. This involves reviewing the code provided by the front-office, running simulation stress tests, and developing alternative implementations as a cross-check. This new dimension brings a fresh set of challenges however, as the validation team must now deal with the varied technologies used by the front-office groups (Excel/VBA, Matlab, C++). This challenge is compounded by the fact that these teams may have a very strong Quant culture, but usually less than a strong IT discipline. If they decide to develop an in-house analytics library, they will often rely on third-party components.

- A still wider interpretation of the role is to validate not only the model implementation, but the whole *process* where the model comes into play:

  – What the model is used for: either getting a delta from a calibrated price for short-term arbitrage, or providing a reference price for an exotic product which is to stay in the book; whether the complex product at hand is hedged in various ways with indicators provided by the model. In fact the right point of view is very often to validate not a model, but a strategy performed by a market actor which involves model-computed quantities at some stage. A typical example in the Vanna-Volga strategy, mostly used on FX markets, which is not a *model* in the traditional sense.

  – The market data and other data sources on which the strategy depends: do you calibrate your models on the right data? For instance there are several different fields in Bloomberg for an option price: do you use the right one? Regarding dividends and repos on equity markets, where does your data come from? How do you handle implied dividends and/or repos stemming from the call/put parity in force on the vast majority of equity indices?

  – At which horizon do you use the model which have been calibrated on 3-years market data? How is the model extrapolated?

  – At which frequency do you re-calibrate your model? How do you handle the discrepancy between the implied volatility/correlation markets and the historical ones?

  – What happens in case of a Credit event for one of the counterparties?

The academic literature has also recognized the issue early on, starting with contributions by Merton and Scholes themselves [18], shortly after the publication of the Black-Scholes model. Academic contributions to the analysis of Model Risk can be organized into three strands of literature. The first strand provides a typology of the risks involved, and has been used by regulators and auditors to define best practice. The second strand focuses on empirical tests of model robustness, while the most recent current attempts to define a measure that would be homogeneous to market risk, and thus could be directly used to define capital requirement. These contributions are summarized in the next 3 sections.

Qualitative Assessment of Model Risk

Derman classification

An early description of Model Risk is due to E. Derman [8]. In this paper, Derman lists 6 types of risk:

1. Inapplicability of modeling: a mathematical model may not be relevant to describe the problem at hand.
2. Incorrect model: the risk of using a model that does not accurately describe the reality being modeled. This is the most common interpretation of Model Risk.
3. Correct model, incorrect solutions, leading, for example, to inconsistent pricing of related claims.
4. Correct model, inappropriate use: the risk related to an inaccurate numerical solution of an otherwise correct model. For example, the risk related to Monte-Carlo calculations with too few simulations.
5. Badly approximated solutions. This risk appears when numerical methods are used to solve a model.
6. Software and hardware bugs.
7. Unstable data. Financial data is of notoriously poor quality. Models must be robust with respect to errors in input data.

Derman's recommendations for coping with model risk are mostly organizational: use a comprehensive approach to model validation, test exhaustively, create good interfaces to minimize user error, etc. This typology of model risk is the foundation of the processes used by auditors and consulting firms to validate the use of models inside financial institutions. It also forms the framework, and provides a series of tests to be used for assessing model risk from a legal perspective [17]. Most importantly, these categories have inspired regulators. In the paper "Supervisory guidance for assessing banks' financial instrument fair value practices" of April 2009 [20], the Basel Committee on Banking Supervision issues recommendations on controlling model and operational risk. The elements of a "rigorous model validation" are described as follows:

> Validation includes evaluations of:
>
> - The model's theoretical soundness and mathematical integrity.
> - The appropriateness of model assumptions, including consistency with market practices and consistency with relevant contractual terms of transactions.
> - Sensitivity analysis performed to assess the impact of variations in model parameters on fair value, including under stress conditions.
> - Benchmarking of the valuation result with the observed market price at the time of valuation or independent benchmark model.

It further states the need to assess a valuation adjustment that will reflect model risk ([21], 718(cxi-1)):

> For complex products including, but not limited to, securitisation exposures and n-th-to-default credit derivatives, banks must explicitly assess the need for valuation adjustments to reflect two forms of model risk: the model risk associated with using a possibly incorrect valuation methodology; and the risk associated with using unobservable (and possibly incorrect) calibration parameters in the valuation model.

Auditors finally use the same categories to describe best-practice guidelines [22].

When everything else fails

One area where the qualitative assessment of model risk is of major importance is that of *very* illiquid assets. This covers most of the structured Credit asset-back-type securities, like ABS and RBMS and their derivatives, as well as hyper-exotic tailor-made hybrid products. In such cases it is already a challenge to identify the various risk factors. Among these the vast majority will be non-tradeable so that the pure model price is the only benchmark. The model parameters will often be very roughly calibrated to dubious-and-not-always-meaningful data.

In such situations, a parsimonious model with economically meaningful parameters is an adequate tool (cf [12]). The qualitative behavior of the product payoff and of the product price with respect to the model parameter is then key to the assessment of Model Risk . More quantitative tools can be used; yet only large simulation engines will produce reliable quantities in many circumstances - the input of which will mostly be stylised, qualitative parameters.

Model risk is not the ultimate clue to illiquidity: especially in that context, Model Risk needs to be assessed as part of a comprehensive risk management policy. In less extreme contexts, the framework outlined above provides a guideline for spelling out compliance requirements, but does not provide a quantitative measure of Model Risk. How to provide such a measure is the topic of the research reviewed in the next section.

Empirical Assessment of Model Risk

There is a large body of literature on the empirical assessment of Model Risk, starting with contributions by Merton and Scholes themselves [18]. The general idea is to simulate the pricing and hedging policy of a derivative product under a number of assumptions in order to provide an empirical measure of the associated risk.

Dynamic hedging simulation

In 1985, Rubinstein [25] used this simulation method to document a major challenge to the Black-Scholes model, which is the empirical observation of a "volatility smile". Among many other publications, we note the paper by Bakshi, Cao and Chen [3] which performs a comprehensive assessment of ten years of development in pricing models for equity derivatives, covering stochastic volatility models, models with jumps and with stochastic interest rate. Interestingly, they find that the most sophisticated model is not the most effective tool for hedging:

> Overall, incorporating stochastic volatility and jumps is important for pricing and internal consistency. But for hedging, modeling stochastic volatility alone yields the best performance.

Also of interest is the range of tests that the authors apply to compare models:

1. Internal consistency of implied parameters/volatility with relevant time-series data,

2. Out-of-sample pricing, and

3. Quality of dynamic hedging.

Note that the validation of a model through dynamic hedging simulation is what Jarrow and Yildirim suggest too in the context of the inflation markets in their celebrated paper [24].

The calibration conundrum

Following the development of the local volatility model, which allows to calibrate all the available vanilla quotes with a single model (which may have a complex functional form), a perfect model calibration to a set of vanilla options has become the expected standard in the industry.

This apparent success, however, still leaves a number of fundamental issues open:

1. The vanilla quotes on a given day are only a part of the available information, which contains at least the *history* of the underlying, and also the history of the vanilla quotes.
2. The vanilla quotes are in theory equivalent to the marginal distributions of the underlying under the market implied probability measure. This said, they contain no information on the *joint* laws of the underlying process, whereas most of the payoffs of exotic structured products will be highly sensitive to functionals related to these joint laws.

The first issue is partially dealt with in mixed historical/market calibration procedures where an historical *prior* is used in the market calibration strategy. Nevertheless, there is plenty of room for research on robust calibration algorithms that exploit the whole information available.

The second issue has been discussed in deep empirical calibration studies, mostly conducted on the equity market. In [27], Schoutens and alii study the relative behavior of several models which calibrate equally well to the vanilla quotes, and show that they yield vastly different results when pricing path dependent or moment derivatives such as variance swaps. The same kind of work has been performed by Dilip Madan and Ernst Eberlein in [11], who have pioneered the class of Sato processes as a natural choice for the joint dynamics underlying the primary asset process.

The need for simulation

The distribution of the Profit and Loss arising from a dynamic hedging simulation, or the range of prices for a structured product within various 'perfectly' calibrated models are of primary interest to empirically assess Model Risk.

The easier it is to perform such simulations, the better. The challenge is to get a re-usable simulation infrastructure, modular enough to design quickly new tests, altogether with a clean reporting. Such a *simulation facility* is already very useful in a purely simulated world alleviated from the technicalities of real-life contracts (with calendaring, market conventions, market data issues): in fact many phenomenon will be more easily isolated, tackled, hopefully understood, in this simplified environment.

Yet it becomes an invaluable tool if it can handle real-life contracts and market data. A true back-testing simulation infrastructure, in so far as it comes *after* the simplified prototyping step, might be the ultimate tool in some sense. Very few systems offer such facilities, at least with the required degree of modularity and transparency.

Regarding back-testing procedures, the pertaining choice of the time interval on which performing the simulation is a crucial matter. Beyond this issue, there will always be a single path available in the history of the underlying; the simulation of a pertaining richer set of paths in a parsimonious manner (bootstrapping) is a field under active study, where Markov Chain Monte Carlo methods give promising results (cf [4], [7]).

Simulation studies provide the most comprehensive assessment of model risk. Yet, they do not address an important requirement of the Basel directive, that is the need for a normalized measure of model risk that is homogeneous to other risk measures. How to create this measure, that would be in currency unit, is a recent but active research topic which is next reviewed.

Quantification of Model Risk

The robustness of Black Scholes

The mis-specification risk of the Black-Scholes model has been well understood since the late 70's: if the actual volatility that is experienced while delta hedging a derivative is lower than the pricing volatility, then the replicating portfolio will under-perform the derivative, and vice-versa. This result was expanded in a rigorous analysis by El Karoui and al. [13]. The authors show that for Call and Puts, and more generally for convex payoffs, the option seller will make a profit as soon as the selling and hedging volatility is larger than the actual *realized* volatility, which can bear any kind of stochasticity. Moreover, the Profit can be computed: it will be proportional to the accumulation of the running option Gamma times the difference of the squared volatilities.

This property has been known in practice for a long time and it has played a crucial role in the practical usage of the Black-Scholes formula. The robustness of Gaussian hedging strategies in a fixed income context has been studied in [1].

Therefore the Model Risk is well understood and quantified in the case of the Black-Scholes pricing/hedging strategies for *convex* payoffs like Calls, Puts, or convex combinations of those.

The seminal work of Avellaneda and Lyons on model uncertainty

This very nice robustness property does not hold in general for exotic or non-convex payoffs (as a basic call-spread). In 2 simultaneous seminal works, Avellaneda et al. [16] and T.Lyons [15] solved the related problem of finding out the cheapest (resp. highest) price of a super-replicating (resp. sub-replicating) strategy *given the fact that the volatility is unknown* - with the assumption that it lies in a given range.

This Uncertain Volatility Model is in some sense the right generalization of the Black-Scholes model, in so far as it extends the previous robustness property to any kind of payoff. This comes to some cost of course: the corresponding ″super price″ solves a (fully) non-linear PDE. Even if can be efficiently solved in a basic trinomial tree algorithm, this non linearity is a fundamental matter and the management in the UVM framework of a book of derivatives is a quite tricky matter.

This largely accounts for the relatively cold reception made to the UVM model by the practitioners, coupled with the fact that the ″super price″ is in general very high as a consequence of the generalized robustness property. Note that it seems that the UVM approach applied to correlation uncertainty (as presented in [26]) is used for pricing purposes in practice.

Even though the UVM ″super price″ may not be pertaining for trading purposes, yet the UVM ″super price″, or more precisely the UVM ″super price″ minus the ″sub price″ is a very pertaining model risk indicator.

Recent proposals and open issues

More recently, the Basel II market risk framework ([21], 718(cxi) and 718(cxii)) explicitly mandates a valuation adjustment, that impacts Tiers I regulatory capital, to account for model risk. Therefore, one needs to design a method for measuring model risk that is both a coherent risk measure, and can be expressed in currency terms. To frame the issue, it is useful to consider the analogy with the VaR method for computing market risk. The calculation of VaR involves two steps:

- The identification of the market risk factors and the estimation of the dynamic of these risk factors: the classical VaR framework assumes a multivariate log-normal distribution for asset prices

- The definition of a risk measure, for example the 99.5% confidence interval for a 10-day holding period.

What would be the equivalent when considering Model Risk? In this case, the risk factors include the risk of model mis-specification (leaving out important sources of risk, mis-specifying the dynamic of the risk factors), and the risk of improper calibration, even though the chosen model may be perfectly calibrated to a set of liquid instruments. The second step involves defining a reasonable family of models over which the risk should be assessed. The family of models is restricted to the models that can be precisely calibrated to a set of liquid instruments. This constraint alone still defines such a large set of models that further restrictions need to be applied. Intuitively, one needs to define a meaningful notion of "distance" between models, in order to define a normalized measure of Model Risk. Recent research is attempting to provide a rigorous answer to these challenging questions. Rama Cont [5] frames the problem as follows.

- Let I be a set of liquid instruments, with $H_{i\in I}$ being the corresponding payoffs, and $C^*_{i\in I}$ the mid-market prices, with $C^*_i \in [C_i^{\text{bid}}, C_i^{\text{ask}}]$.
- Let $\mathcal{Q}$ be a set of models, consistent with the market prices of benchmark instruments:

$$Q \in \mathcal{Q} \Rightarrow E^Q[H_i] \in [C_i^{\text{bid}}, C_i^{\text{ask}}],\ \forall i \in I \tag{1}$$

Define next the upper and lower price bounds over the family of models, for a payoff X:

$$\overline{\pi} = \sup_{j=1,\dots,n} E^{Q_j}[X] \quad \underline{\pi} = \inf_{j=1,\dots,n} E^{Q_j}[X]$$

The risk measure is finally the range of values caused by model uncertainty:

$$\mu_{\mathcal{Q}} = \overline{\pi}(X) - \underline{\pi}(X)$$

Note the analogy with the UVM framework: in fact the UVM Model Risk indicator sketched in the previous section perfectly fits this framework, with $\mathcal{Q}$ the set of models with a volatility in a given range. The crucial aspect of this procedure is the definition of a suitable set $\mathcal{Q}$. There are many ways of defining it:

- Choose a class of models, and construct a set of models by varying some unobservable parameters, while ensuring that each model calibrates to the set of benchmark instruments.

- Select several types of model (local volatility, stochastic volatility, etc), and calibrate each model to the same set of benchmark instruments.

The first approach is followed by Dumont and Lunven [10]. They tested the method on basket options, priced in a multivariate log-normal framework. The model is calibrated to vanilla options on each element of the basket. This leaves the correlation undetermined. The authors compute an historical distribution of this correlation matrix, and sample it to construct the set $\mathcal{Q}$. The optimization is finally performed heuristically.

The second approach is followed in [27], [11] where numerous mainstream academic equity models are dealt with. As expected [9], model mis-specification induces more variability than calibration error.

It is clear that the variability of models forming the set $\mathcal{Q}$ needs to be normalized. In the same way as one computes "99.5% VaR for a 10 day holding period", one needs a normalizing factor to qualify model risk. In other words, one needs to define the aforementioned notion of "distance" between models. A possible approach could be found in [2],[6] and [19], where the relative entropy between two processes is used to regularize the calibration problem. For payoffs that are not path dependent, one can measure the relative entropy of the two distributions at expiry. This provides a measure of distance that is independent of the nature of the underlying process.

Contrasting with the above analytical approach, Kerkhof and al. [14] develop a method for accounting for Model Risk, using a data-driven method: They estimate an empirical distribution of model outcome, using historical data. The same information is used to compute market risk and model risk, and model risk is expressed as an adjustment (multiplicative factor) to the VaR estimate.

As this brief survey suggests, the quantification of Model Risk is under active study. As with the other sources of risk, the crucial issue is to define a normalized measure that will enable a comparison among exotic products, and ultimately among asset classes and financial institutions.

Conclusion

Model validation encompasses a variety of disciplines and issues. The role of model validation teams has been recognized to be crucial in the recent turmoil. They can tackle at the same time the validation of well-defined components and the validation of a whole business process and meanwhile contribute to a much better understanding of the risk profile of the strategies conducted by the front office.

Under the pressure of the regulators, Model Validation teams will be asked to compute standardized Model Risk indicators, spanning all asset classes, in a documented and reproducible manner. This is a significant technological challenge. To meet this challenge, the model validation teams will need to be equiped with a technology platform that includes:

- Model libraries that can be validated independently of the librairies created by the front office (i.e. developed independently by the Model Validation team, sourced from third party vendors, or both)
- A simulation framework that enables to quickly test new models, payoffs and strategies in a standardized and reproducible manner.

More broadly, the derivatives industry will ultimately need to develop a set of standardized methods for measuring Model Risk. Given the complexity of such measures, their credibility will require an independent calculation framework with the ability to provide *transparent*, *reproducible* and *auditable* results, for internal Model Validation teams, auditors and regulators. This will ease the way towards the practice of meaningful prices and sane risk policies with a quantified impact of model uncertainty.

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